# Implementation of Optimal Monetary Policy

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R ecently the study of optimal monetary policy has shifted from an analysis of the welfare effects of simple parametric policy rules to the solution of optimal planning problems. Both approaches evaluate the welfare effects of monetary policy in an explicit monetary model of the economy, but they differ in the scope of analysis. The first approach is more restrictive in that it finds the optimal policy within a class of prespecified policy rules for the monetary policy instrument. On the other hand, the second approach finds the optimal monetary policy among all allocations that are consistent with a competitive equilibrium in the monetary economy. Since monetary policy, in general, does not choose the economy's allocation but implements policy through a rule for the policy instruments, it is natural to ask whether the policy rule implied by the solution to the planning problem implements the optimal planning allocation. In most work on optimal planning problems, it is indeed taken for granted that the solution of the planning problem can be implemented through some policy rule for the monetary policy instrument but, as we show in this article, this need not always be the case.

There is a vast literature on optimal monetary policy that studies the solution to planning problems. The environments examined are diverse, ranging from models in which there are no private sector distortions other than an inflation tax to models where economies are subject to various types of nominal rigidities. The policymaker is assumed to choose among all the allocations that are consistent with a market equilibrium in the given environment. In addition, different assumptions are made as to whether a policymaker can or cannot commit to his future choices. Under a full-commitment policy, we as-

sume that the policymaker chooses all current and future actions in an initial period. Alternatively, under time consistency we assume that in every period a policymaker chooses the optimal action, taking past outcomes as given. For either specification, the solution to the planning problem specifies a rule that determines the allocation, and part of the allocation is the setting of the policy instrument.

The question is whether the policy rule implied by the solution to the planning problem (or a variation thereof) can implement the optimal allocation for the planning problem. Specifically, how would the competitive economy behave if the monetary authority simply announced the policy rule implied by the solution to the planning problem? In particular, conditional on the policy rule, will there be a unique competitive equilibrium?

Giannoni and Woodford (2002a, 2002b) discuss the implementability of optimal policy for local approximations of the planning problem with full commitment. This starts with a log-linear approximation around the steady state of the solution to the full-commitment problem. Within the approximation framework, implementability of the optimal policy rule is equivalent to the existence and uniqueness of rational expectations equilibria in linear models. As such, implementability is concerned with "dynamic" uniqueness, that is, the existence of a unique stochastic process that characterizes the competitive equilibrium.

King and Wolman (2004) discuss the implementation of Markov-perfect policy rules for time-consistent solutions to the planning problem. King and Wolman (2004) show that Markov-perfect policies with an optimal nominal money stock instrument can imply equilibrium indeterminacy at two levels. First, it can imply multiple steady states. Second, around each steady state it can imply static price level indeterminacy, that is, conditional on future outcomes there can be multiple current equilibrium prices.

In this article, we review implementability of both the optimal full-commitment and time-consistent Markov-perfect monetary policies when the policymaker uses a nominal money stock instrument. We study optimal policy in a simple New Keynesian economic model as described in Wolman (2001) and King and Wolman (2004). We first characterize the solution to a linearized version of the first-order conditions (FOCs) of the planning problems. We show that optimal monetary policy locally implements the planning allocation for the full-commitment and the Markov-perfect case. We then study whether the policy rules implement the planning allocations globally. We review King and Wolman's (2004) argument that the Markov-perfect policy rule cannot implement the planning allocation. Finally, we provide a partial argument that the full-commitment policy rule globally implements the planning allocation.

## 1. A SIMPLE ECONOMY WITH STICKY PRICES

We investigate the question of the implementability of optimal monetary policy within the confines of a simple New Keynesian economic model. The model contains an infinitely lived representative household with preferences over consumption and leisure. The consumption good is produced using a constant-returns-to-scale technology with a continuum of differentiated intermediate goods. Each intermediate good is produced by a monopolistically competitive firm with labor as the only input. Intermediate goods firms set the nominal price for their products for two periods, and an equal share of intermediate firms adjust their nominal price in any period. We describe a symmetric equilibrium for the economy, and we characterize the two distortions that make the equilibrium allocation suboptimal relative to the Pareto-optimal allocation.

### The Representative Household

The representative household's utility is a function of consumption, $c_t$, and the fraction of time spent working, $n_t$,

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[ \ln c_t - \chi n_t \right], \tag{1}$$

where $\chi \geq 0$, and $0 < \beta < 1$. The household's period budget constraint is

$$P_t c_t + B_{t+1} + M_t \leq W_t n_t + R_{t-1} B_t + M_{t-1} + D_t + T_t, \tag{2}$$

where $P_t$ ($W_t$) is the money price of consumption (labor), $B_{t+1}$ ($M_{t+1}$) are the end-of-period holdings of nominal bonds (money), $R_{t-1}$ is the gross nominal interest rate on bonds, $T_t$ are lump-sum transfers, and $D_t$ is profit income from firms owned by the representative household. The household is assumed to hold money in order to pay for consumption purchases

$$M_t = P_t c_t. \tag{3}$$

We will use the term "real" to denote nominal variables deflated by the price of consumption goods, and we use lower-case letters to denote real variables. For example, real balances are $m \equiv M/P$.

The FOCs of the representative household's problem are

$$\chi = w_t/c_t, \text{ and} \tag{4}$$

$$1 = \beta E_t \left[ \frac{c_t}{c_{t+1}} \cdot \frac{R_t}{P_{t+1}/P_t} \right]. \tag{5}$$

Equation (4) states that the marginal utility derived from the real wage equals the marginal disutility from work. Equation (5) is the Euler equation, which states that if the real rate of return increases, then the household increases future consumption relative to today's consumption.

**Firms**

The consumption good is produced using a continuum of differentiated inter-
mediate goods as inputs to a constant-returns-to-scale technology. Producers
of the consumption good behave competitively in their markets. There is a
measure one of intermediate goods, indexed $j \in [0, 1]$. Production of the
consumption good $c$ as a function of intermediate goods, $y(j)$, used is

$$c_t = \left[ \int_0^1 y_t(j)^{(\varepsilon-1)/\varepsilon} dj \right]^{\varepsilon/(\varepsilon-1)}, \tag{6}$$

where $\varepsilon > 1$. Given nominal prices, $P(j)$, for the intermediate goods, the
nominal unit cost and price of the consumption good is

$$P_t = \left[ \int_0^1 P_t(j)^{1-\varepsilon} dj \right]^{1/(1-\varepsilon)}. \tag{7}$$

For a given level of production, the cost-minimizing demand for intermediate
good $j$ depends on the good's relative price, $p(j) \equiv P(j)/P$,

$$y_t(j) = p_t(j)^{-\varepsilon} c_t. \tag{8}$$

Each intermediate good is produced by a single firm, and $j$ indexes both
the firm and good. Firm $j$ produces $y(j)$ units of its good using a constant-
returns technology with labor as the only input,

$$y_t(j) = \xi_t n_t(j), \tag{9}$$

and $\xi_t$ is a positive iid productivity shock with mean one. Each firm behaves
competitively in the labor market and takes wages as given. Real marginal
cost in terms of consumption goods is

$$\psi_t = w_t/\xi_t. \tag{10}$$

Since each intermediate good is unique, intermediate goods producers
have some monopoly power, and they face downward sloping demand curves,
(8). Intermediate goods producers set their nominal price for two periods, and
they maximize the discounted expected present value of current and future
profits:

$$\max_{P_t(j)} \left( \frac{P_t(j)}{P_t} - \psi_t \right) y_t(j) + \beta E_t \left[ \frac{c_t}{c_{t+1}} \cdot \left( \frac{P_t(j)}{P_{t+1}} - \psi_{t+1} \right) y_{t+1}(j) \right]. \tag{11}$$

Since the firm is owned by the representative household, the household's
intertemporal marginal rate of substitution is used to discount future profits.
Using the definition of the firm's demand function, (8), the first-order condition
for profit maximization can be written as

$$0 = \left(\frac{P_t(j)}{P_t}\right)^{1-\varepsilon}\left(1 - \mu\frac{\psi_t}{P_t(j)/P_t}\right) \tag{12}$$

$$+\beta E_t\left[\left(\frac{P_t(j)}{P_{t+1}}\right)^{1-\varepsilon}\left(1 - \mu\frac{\psi_{t+1}}{P_t(j)/P_{t+1}}\right)\right],$$

with $\mu = \varepsilon/(\varepsilon - 1)$.

## A Symmetric Equilibrium

We will assume a symmetric equilibrium, that is, all firms who face the same constraints behave the same. Each period, half of all firms have the option to adjust their nominal price. This means that in every period there will be two firm types: the firms who adjust their nominal price in the current period, type 0 firms with relative price $p_0$, and the firms who adjusted their price in the last period, type 1 firms with current relative price $p_1$.

Conditional on a description of monetary policy, the equilibrium of the economy is completely described by the sequence of marginal cost, relative prices, inflation rates, nominal interest rates, aggregate output, and real balances $\{\psi_t, p_{0,t}, p_{1,t}, \pi_t, R_t, c_t, m_t\}$ such that (3), and

$$\psi_t = \chi c_t/\xi_t, \tag{13}$$

$$1 = \frac{1}{2}\left[p_{0,t}^{1-\varepsilon} + p_{1,t}^{1-\varepsilon}\right], \tag{14}$$

$$0 = p_{0,t}^{1-\varepsilon}\left(1 - \mu\frac{\psi_t}{p_{0,t}}\right) + \beta E_t\left[p_{1,t+1}^{1-\varepsilon}\left(1 - \mu\frac{\psi_{t+1}}{p_{1,t+1}}\right)\right], \tag{15}$$

$$\pi_{t+1} = \frac{p_{0,t}}{p_{1,t+1}}, \text{ and} \tag{16}$$

$$1 = \beta E_t\left[\frac{c_t}{c_{t+1}} \cdot \frac{R_t}{\pi_{t+1}}\right]. \tag{17}$$

Equation (13) uses the optimal labor supply condition (4) in the definition of marginal cost (10). Equation (14) is the price index equation (7) and equation (15) is the profit maximization condition (12) for the two firm types. Equation (16) just restates how next period's preset relative price $p_{1,t+1}$ is related to the relative price that is set in the current period, $p_{0,t}$, through the inflation rate $\pi_{t+1}$. Finally, equation (17) is the household's Euler equation, (5).

## Distortions

Allocations in this economy are not Pareto-optimal because of two distortions. The first distortion results from the monopolistically competitive structure of intermediate goods productions: the price of an intermediate good is not equal to its marginal cost. The average markup in the economy is the inverse of the

real wage, $P_t/W_t$, that is, according to equation (10), the inverse marginal cost, $1/(\xi_t \psi_t)$. The second distortion reflects inefficient production when relative prices are different from one. Using the expressions for the production of final goods and the demand functions for intermediate goods, (6) and (8), we can obtain the total demand for labor as a function of relative prices and aggregate output. Solving aggregate labor demand for aggregate output, we obtain an "aggregate" production function

$$d_t c_t = \xi_t n_t \text{ with } d_t \equiv (1/2)\left(p_{0,t}^{-\varepsilon} + p_{1,t}^{-\varepsilon}\right). \tag{18}$$

Given the symmetric production structure, equations (6) and (9), efficient production requires that equal quantities of each intermediate good are produced. Allocational efficiency is reflected in the term $d_t \geq 1$. The allocation is efficient if $p_{0,t} = p_{1,t} = d_t = 1$.

For the following analysis of optimal policy, it is useful to rewrite the household's period utility from the equilibrium allocation as a "reduced form" utility function of the markup and efficiency distortion. Combining expression (13) for equilibrium consumption as a function of marginal cost and productivity with the characterization of the aggregate production function (18) yields equilibrium work effort

$$n_t = d_t \psi_t / \chi. \tag{19}$$

We can substitute expressions (13) and (19) for consumption and work effort in the household's utility function and obtain the reduced form utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln\left(\psi_t\right) - d_t \psi_t\right], \tag{20}$$

after dropping any constant or additive exogenous terms.

## 2.   MONETARY POLICY

Since the allocation of the above-described monopolistically competitive equilibrium with sticky prices is suboptimal, there is the potential for welfare-improving policy interventions. In view of the role of nominal rigidities, we want to characterize optimal monetary policy. In particular, we want to know how optimal monetary policy can be implemented given some choice of policy instrument. We examine the implications of choosing the nominal money stock as the policy instrument. This is the policy instrument considered in King and Wolman (2004), where they assume that the policymaker chooses a sequence for the nominal money stock $\{M_t\}$. Alternatively the policymaker could select the nominal interest rate, $R_t$, as the policy instrument. The choice of policy instrument can be crucial for questions of the implementability of optimal monetary policy, and we will get back to this issue in the conclusion.

For the analysis of the monetary policy planning problem, it is convenient to define monetary policy in terms of the money stock normalized relative to

the preset nominal prices,

$$m_{1t} = \frac{M_t}{P_{1,t}},$$  (21)

rather than the nominal money stock, $M_t$, directly. This normalization is not restrictive for the analysis of a policymaker that can commit to future policy choices, the full-commitment case. In the case of time-consistent policies, when a policymaker cannot commit to future policy choices, we will argue that for the particular class of Markov-perfect policies that we study, the normalized money stock is the relevant choice variable. Combining the policy rule with the cash-holding condition, (3), and using $P_{1,t} = P_{0,t-1}$, we obtain an equilibrium condition for consumption

$$c_t = p_{1,t}m_{1t}.$$  (22)

## Optimal Monetary Policy

The objective of monetary policy is well-defined: the policymaker is to choose an allocation that maximizes the representative household's utility subject to the constraint that the allocation can be supported as a competitive equilibrium. For our simple example, any allocation that satisfies equations (13)–(16), (18), and (22) is a competitive equilibrium. We summarize these constraints as

$$E_t \left[ h \left( x_{t+1}, x_t; \xi_{t+1}, \xi_t \right) \right] = 0 \text{ for } t \geq 0.$$  (23)

The vector $x_t = (y_t, z_t)$ contains the private sector variables, $y_t = (p_{0,t}, p_{1,t}, \pi_t, \psi_t, d_t, c_t)$, and the policy instrument, $z_t = m_{1t}$.[1] Formally, the policymaker's optimization problem is then defined as

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(x_t) \text{ s.t. } E_t \left[ h \left( x_{t+1}, x_t; \xi_{t+1}, \xi_t \right) \right] = 0 \text{ for } t \geq 0,$$  (24)

where $u$ denotes the period utility function of the representative household as defined in equation (20). A solution to this problem will have $x_t$ as a function of the current and past state of the economy.

We will solve two alternative versions of the planning problem. First, we assume that the policymaker at time zero chooses once and for all the optimal allocation among all feasible allocations that can be supported as a competitive equilibrium. This approach delivers the constrained optimal allocation, but frequently the chosen allocation is not time consistent. The allocation is not time consistent in the sense that if a policymaker gets the option to reconsider his choices after some time, he would want to deviate from the initially chosen

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[1] The characterization of the private sector involves equilibrium prices and quantities. With some abuse of standard terminology, we will call the vector $y$ the equilibrium allocation.

path. The alternative approach then finds optimal time-consistent monetary policies. In particular, we will restrict attention to Markov-perfect policy rules, that is, rules that make policy choices contingent on payoff-relevant state variables only.

For the planning problem, we are not specific about how the policymaker can implement the policy: we simply assume that the policymaker can select any allocation subject to the constraint that the allocation is consistent with a competitive equilibrium allocation. We will say that a policy can be implemented if a unique rational expectations equilibrium exists when the policymaker sets the policy instrument, $z_t$, according to the state-contingent rule implied by the planning problem.

### *Optimal Policy with Full Commitment*

Suppose that at time zero the policymaker chooses a sequence $\{x_t\}$ for the market allocation and the policy instrument that solves problem (24). We assume that the policymaker is committed to this outcome for all current and future values of the market outcome and the instrument. The FOCs for this constrained maximization problem are

$$
\begin{aligned}
0 &= Du\left(x_t\right) + \lambda_t E_t \left[D_2 h\left(x_{t+1}, x_t; \xi_{t+1}, \xi_t\right)\right] \\
&\quad + \lambda_{t-1} D_1 h\left(x_t, x_{t-1}; \xi_t, \xi_{t-1}\right) \text{ for } t > 0, \text{ and}
\end{aligned}
\tag{25}
$$

$$
0 = Du\left(x_t\right) + \lambda_t E_t \left[D_2 h\left(x_{t+1}, x_t; \xi_{t+1}, \xi_t\right)\right] \text{ for } t = 0.
\tag{26}
$$

Note that the FOC for the initial time period, $t = 0$, is essentially the same as the FOCs for future time periods, $t > 0$, if we assume that the lagged Lagrange multiplier in the initial time period is zero, $\lambda_{-1} = 0$. This simply means that in the initial time period, the policymaker's choices are not constrained by past market expectations of outcomes in the initial period.

Marcet and Marimon (1998) show how to rewrite the planning problem as a recursive saddlepoint problem such that dynamic programming techniques can be applied. Following their approach, the Lagrange multiplier, $\lambda_{t-1}$, can be interpreted as a state that reflects the past commitments of the planner. Given the dynamic programming formulation, the optimal policy choice will then be a function of the state of the economy,

$$
x_t = g_x^{FC}\left(\lambda_{t-1}, \xi_t\right) \text{ and } \lambda_t = g_\lambda^{FC}\left(\lambda_{t-1}, \xi_t\right).
\tag{27}
$$

The policymaker's optimization problem is not time consistent because of the particular status of the initial period. If a policymaker gets the opportunity to reevaluate his choices at some time $t' > 0$, then equation (25) will no longer characterize the optimal decision at $t'$. Rather equation (26) will apply at the time $t'$, and, in general, the policymaker would want to deviate from his original decision. If the policymaker has no way to precommit to future policy actions, the optimal policy will therefore not be time consistent.

### Markov-Perfect Optimal Policy

We study a particular class of time-consistent policies, namely Markov-perfect policies. For a Markov-perfect policy, the optimal policy rule is restricted to depend on payoff-relevant state variables only, that is, predetermined variables that constrain the attainable allocations of the economy. We can think of today's policymaker as taking his own future actions as given by a policy rule that makes his choices contingent on the future payoff-relevant state variables. Given these future choices, the policymaker's optimal choice for today will then also depend on payoff-relevant state variables only.

In our environment, predetermined nominal prices do not constrain the policymakers' choices among the allocations that are consistent with a competitive equilibrium. Even though the nominal price set by a firm that adjusted its price in the last period, $P_{1,t}$, is predetermined, the relevant variable is that firm's relative price, $p_{1,t}$, which is not predetermined. Since the predetermined nominal price is not payoff-relevant, the policymaker has to choose the nominal money stock in a way such that the predetermined nominal price cannot affect outcomes. But this just means that the policymaker cannot choose the nominal money stock, $M_t$, but has to choose the normalized money stock, $m_{1t}$.

Our environment as described by (23) then has the feature that, except for the exogenous shocks, $\xi_t$, there are no predetermined variables that constrain the equilibrium allocation. In other words, in any time period the values for the variables that characterize the competitive equilibrium have to be consistent with future values of the same variables, but the variables can be chosen independently of any values they took in the past.

In a Markov-perfect equilibrium, the current policymaker then assumes that future choices and outcomes are time invariant functions of $\xi$, $x_{t'} = g_x^{MP}(\xi_{t'})$, for $t' > t$. For this reason, current policy choices have no effect on future outcomes, and the policymaker's choice problem simplifies to

$$x_t^* \left( \xi_t; g_x^{MP} \right) \quad = \quad \arg \max_x u(x_t) \tag{28}$$
$$\text{s.t. } 0 = E_t \left[ h \left( g_x^{MP} \left( \xi_{t+1} \right), x_t; \xi_{t+1}, \xi_t \right) \right].$$

The FOCs for this problem coincide with the FOCs of the optimization problem with commitment for the initial period, equation (26).[2] In a time-consistent Markov-perfect equilibrium, the optimal policy choice satisfies $x_t^* \left( \xi_t; g_x^{MP} \right) = g_x^{MP} \left( \xi_t \right)$.

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[2] In general, the FOCs for a Markov-perfect optimal policy are different from the initial period FOCs for an optimal policy with full commitment. If there are endogenous state variables, then even with Markov-perfect optimal policies, a policymaker can influence future policy choices by changing next period's state variables and thereby affecting the constraint set of next period's policymaker.

**Implementability of Optimal Policy**

If the only requirement for feasible monetary policy is the consistency with a competitive equilibrium, then there is no reason to distinguish between private sector choices, $y_t$, and the policy instrument, $z_t$. We might as well assume that the policymaker chooses both variables, $x_t$, subject to the consistency requirements. Now suppose that the outcome of the optimization problem is a policy rule that specifies choices for the instrument and the private sector allocation contingent on outcomes that may include the current and past states of the economy

$$z_t = g_{zt}(\cdot) \text{ and } y_t = g_{xt}(\cdot). \tag{29}$$

A somewhat narrower definition of what constitutes a feasible monetary policy not only requires that the allocations implied by $g$ are consistent with a competitive equilibrium, but also requires that, conditional on the rule for the policy instrument, $g_z$, the rule for the private sector allocation, $g_y$, is the unique competitive equilibrium outcome. That is, $g_y$ is the unique solution of

$$E_t \left[ h \left( y_{t+1}, g_{z,t+1}(\cdot), y_t, g_{zt}(\cdot); \xi_{t+1}, \xi_t \right) \right] = 0 \text{ for } t \geq 0. \tag{30}$$

If we cannot find a unique solution, $g_y$, to this dynamic system, then we say that the optimal policy cannot be implemented since the associated competitive equilibrium is indeterminate.

In the case of full-commitment policy rules, we can consider an expanded version of the planner's policy rule. Suppose that the planner can respond contemporaneously to deviations of the competitive equilibrium allocation from the allocation implied by the full-commitment policy rule. Then we can define a modified rule for the policy instrument

$$\tilde{g}_z^{FC} \left( y_t, \lambda_{t-1}, \xi_t \right) = g_z^{FC} \left( \lambda_{t-1}, \xi_t \right) + H \left[ y_t - g_{y,t}^{FC} \left( \lambda_{t-1}, \xi_t \right) \right],$$

where $H(0) = 0$. Since the choice of the function $H$ is arbitrary, except for the origin normalization, it then appears that, under these circumstances, a planner can always implement the full-commitment solution. Note that a Markov-perfect policy rule cannot be augmented in this way since the contemporaneous private sector allocation is not a payoff-relevant state variable.

## 3.    LOCAL PROPERTIES OF OPTIMAL POLICY

We now discuss the local dynamics of full-commitment and Markov-perfect optimal policy for our simple economy from Section 1. We derive necessary conditions for the optimal policy and characterize the deterministic steady state of the economy for the types of policy. We then study the properties of optimal policy for a local approximation around its steady state. Our approach follows King and Wolman (1999) and Khan, King, and Wolman (2003) in that we study the dynamics of a linear approximation to the FOCs and constraints

of the optimal planning problem.[3] The two optimal policies imply different policy rules for a money stock instrument. We show that for the local approximation, both implied that policy rules implement a unique rational expectations equilibrium.

Consider a policymaker who uses the money supply as an instrument, that is, the policymaker chooses the money stock according to equation (21). We can then write the competitive equilibrium conditional on the instrument choice in terms of the variables $y_t = p_{1,t}$ and $z_t = m_{1t}$. Conditional on the relative preset price and the policy instrument, consumption is determined by (22); the relative flexible price is determined by (14); allocational efficiency is determined by (18); and marginal cost is determined by (13) and (22). The nominal interest rate is determined residually from equation (17). The policymaker's objective function is

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \ln\left(m_{1t} p_{1,t}\right) - \chi d\left(p_{1,t}\right) m_{1t} p_{1,t} / \xi_t \right\}, \tag{31}$$

and the FOC for profit maximization (15) corresponds to the dynamic constraint (23) for $t \geq 0$:

$$\begin{aligned}
0 = {} & p_0\left(p_{1,t}\right)^{1-\varepsilon}\left(1 - \mu\chi \frac{p_{1,t}}{p_0\left(p_{1,t}\right)} \frac{m_{1t}}{\xi_t}\right) \\
& + \beta E_t\left[p_{1,t+1}^{1-\varepsilon}\left(1 - \mu\chi \frac{m_{1t+1}}{\xi_{t+1}}\right)\right].
\end{aligned} \tag{32}$$

## Optimal Policy with Full Commitment

Under full commitment, the policymaker maximizes the value function (31) subject to the constraints (32). The FOCs corresponding to equations (25) for $t > 0$ are

$$0 = \frac{1}{m_{1t}/\xi_t} - \chi d_t p_{1t} - \mu\chi\lambda_t p_{ot}^{-\varepsilon} p_{1t} - \mu\chi\lambda_{t-1} p_{1t}^{1-\varepsilon}, \text{ and} \tag{33}$$

$$0 = \frac{1}{p_{1t}} - \chi \frac{m_{1t}}{\xi_t} d_t - \chi \frac{m_{1t}}{\xi_t} p_{1t} \frac{\partial d_t}{\partial p_{1t}} \tag{34}$$

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$$+\lambda_t p_{0t}^{-\varepsilon}\left\{(1-\varepsilon)\left(1-\mu\chi\frac{m_{1t}}{\xi_t}\frac{p_{1t}}{p_{0t}}\right)\frac{\partial p_{0t}}{\partial p_{1t}}-\mu\chi\frac{m_{1t}}{\xi_t}\left(1-\frac{p_{1t}}{p_{0t}}\frac{\partial p_{0t}}{\partial p_{1t}}\right)\right\}$$

$$+\lambda_{t-1}p_{1t}^{-\varepsilon}(1-\varepsilon)\left(1-\mu\chi\frac{m_{1t}}{\xi_t}\right).$$

Equation (33) denotes the FOC with respect to real balances, $m_1$, and equation (34) denotes the FOC with respect to the relative price, $p_1$.

### The Deterministic Steady State of the Full-Commitment Policy

In the deterministic steady state of the full-commitment policy, there is zero inflation (King and Wolman 1999; Wolman 2001). We can easily verify that $\pi = p_0 = p_1 = d = 1$ is indeed a deterministic steady state of equations (32), (33), and (34). Combining equation (13) with the monetary policy equation (22) yields an expression that relates marginal cost to real balances and the preset relative price

$$\psi = \chi m_1 p_1. \tag{35}$$

We can substitute this expression for marginal cost in the FOC for profit maximization of price-adjusting firms, (32), and, using the definition of the inflation rate $\pi$, (16), obtain

$$m_1 = \frac{1}{\chi\mu}\frac{1+\beta\pi^{\varepsilon-1}}{1/\pi+\beta\pi^{\varepsilon-1}}. \tag{36}$$

Thus conditional on no inflation, $\pi^{FC} = 1$, real balances are $m_1^{FC} = 1/(\chi\mu)$, and marginal cost is $\psi^{FC} = 1/\mu$. Substituting for marginal cost in the FOC for real balances (33) yields the steady state value for the Lagrange multiplier $\lambda^{FC} = (1 - 1/\mu)/2$, and the FOC for preset relative prices, (34), is satisfied.

### Local Properties of the Full-Commitment Solution

First, we show that the solution to the full-commitment problem stabilizes the prices in response to productivity shocks (King and Wolman 1999). Second, we show that the full-commitment policy rule implements the competitive equilibrium. In the following, let a hat denote the percentage deviation of a variable from its steady state value.

The log-linear approximation of equations (32), (33) and (34) around the no-inflation steady state for $t > 0$ are

$$0 = 2\hat{p}_{1t}+\left(\hat{m}_{1t}-\hat{\xi}_t\right)+\beta E_t\left[\hat{m}_{1,t+1}-\hat{\xi}_{t+1}\right], \tag{37}$$

$$0 = \hat{p}_{1t}+\left(\hat{m}_{1t}-\hat{\xi}_t\right)+\lambda^{FC}\left(\hat{\lambda}_t+\hat{\lambda}_{t-1}\right), \text{ and} \tag{38}$$

$$0 = \left[\mu\frac{2\mu-1}{\mu-1}\right]\hat{p}_{1t}+[1+\chi(\mu-1)]\left(\hat{m}_{1t}-\hat{\xi}_t\right)+(\mu-1)\hat{\lambda}_t. \tag{39}$$

We solve this linear difference equation system through the method of undetermined coefficients. Given the structure of the equation system, it is reasonable to guess that the only relevant state variable is the lagged Lagrange multiplier, $\lambda_{t-1}$, and that the solution is of the form

$$\hat{m}_{1t} - \hat{\xi}_t = \gamma \hat{\lambda}_{t-1}, \ \hat{p}_{1t} = \theta \hat{\lambda}_{t-1}, \text{ and } \hat{\lambda}_t = \omega \hat{\lambda}_{t-1} \text{ for } t > 0. \quad (40)$$

Now substitute these expressions in equations (37)–(39) and confirm that they solve the difference equation system. This procedure yields three equations that can be solved for the unknowns $(\omega, \gamma, \rho)$.

The optimal full-commitment policy increases normalized real balances $m_1$ with productivity shocks such that relative prices are not affected, (40). Relative prices respond to past commitments of the policymaker as reflected in the Lagrange multiplier $\lambda$, and the Lagrange multiplier evolves independently of productivity shocks. When the Lagrange multiplier attains its steady state value it stays there and optimal policy from thereon fixes the price level and relative prices. We do not prove it, but for reasonable numerical values of $(\beta, \mu, \chi)$ the coefficient $\omega$ is negative but less than one in absolute value, that is, the system oscillates, but it is stable. In Figure 1 we graph the transitional dynamics of the economy for some parameter values that are standard for quantitative economic analysis, $\beta = 0.99$, $\mu = 1.1$, and $\chi = 1$. As we can see, all variables display dampened oscillations around their steady state values. As discussed above, the FOCs for the initial period of the full-commitment problem are equivalent to the FOCs (38) and (39) with $\lambda_{-1} = 0$, that is, $\hat{\lambda}_{-1} = -1$. Thus during a transition period, as the Lagrange multiplier converges to its steady state value, relative prices change in proportion to the value of the Lagrange multiplier.
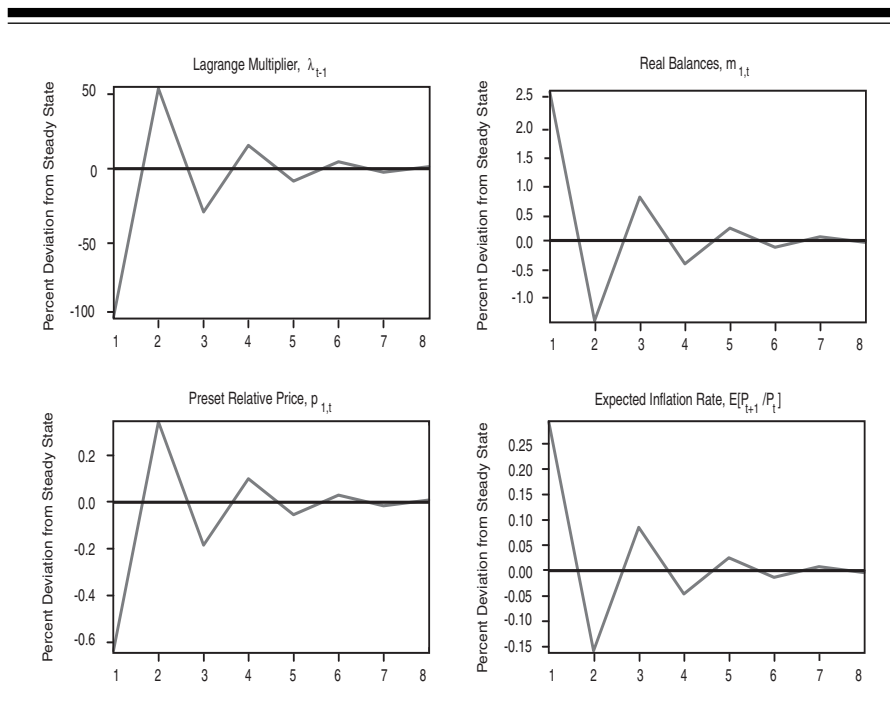
The money-supply policy rule, defined as the first and third expression in (40), implements the optimal allocation as a competitive equilibrium. To see this, substitute the policy rule into the log-linear approximation of the optimal pricing equation (37), and we get

$$\hat{p}_{1t} = \frac{1}{2} \gamma \left(1 + \beta \omega\right) \hat{\lambda}_{t-1}. \quad (41)$$

Thus, conditional on the full-commitment optimal policy rule for real balances, there exists a unique rational expectations equilibrium (REE) for the economy.

**Markov-Perfect Optimal Policy**

For a Markov-perfect optimal monetary policy, the policymaker at time $t$ maximizes the value function (31) subject to the constraints (32), assuming that future policy choices are some function of the future exogenous shock. The FOCs for this problem correspond to equations (26) for $t = 0$ and are

**Figure 1**



Lagrange Multiplier, $\lambda_{t-1}$

Real Balances, $m_{1,t}$

Preset Relative Price, $p_{1,t}$

Expected Inflation Rate, $E[P_{t+1}/P_t]$

$$0 = \frac{1}{m_{1t}/\xi_t} - \chi d_t p_{1t} - \mu \chi \lambda_t p_{ot}^{-\varepsilon} p_{1t}, \text{ and} \tag{42}$$

$$0 = \frac{1}{p_{1t}} - \chi \frac{m_{1t}}{\xi_t} d_t - \chi \frac{m_{1t}}{\xi_t} p_{1t} \frac{\partial d_t}{\partial p_{1t}} \tag{43}$$

$$+ \lambda_t p_{0t}^{-\varepsilon} \left\{ (1-\varepsilon) \left( 1 - \mu \chi \frac{m_{1t}}{\xi_t} \frac{p_{1t}}{p_{0t}} \right) \frac{\partial p_{0t}}{\partial p_{1t}} - \mu \chi \frac{m_{1t}}{\xi_t} \left( 1 - \frac{p_{1t}}{p_{0t}} \frac{\partial p_{0t}}{\partial p_{1t}} \right) \right\}.$$

Equation (42) denotes the FOC with respect to real balances, $m_1$, and equation (43) denotes the FOC with respect to the relative price, $p_1$.

### The Deterministic Steady State of the Markov-Perfect Policy

The deterministic steady state of the Markov-perfect equilibrium has positive inflation, as opposed to the steady state of the full-commitment solution. It is straightforward to show that optimal policy does not stabilize prices in the steady state. Suppose to the contrary that there is no inflation in the steady state, $p_0 = p_1 = 1$, then evaluating equations (32), (42), and (43) at their deterministic steady state implies that $\partial d / \partial p_1 < 0$. But with stable prices,

**Figure 2**
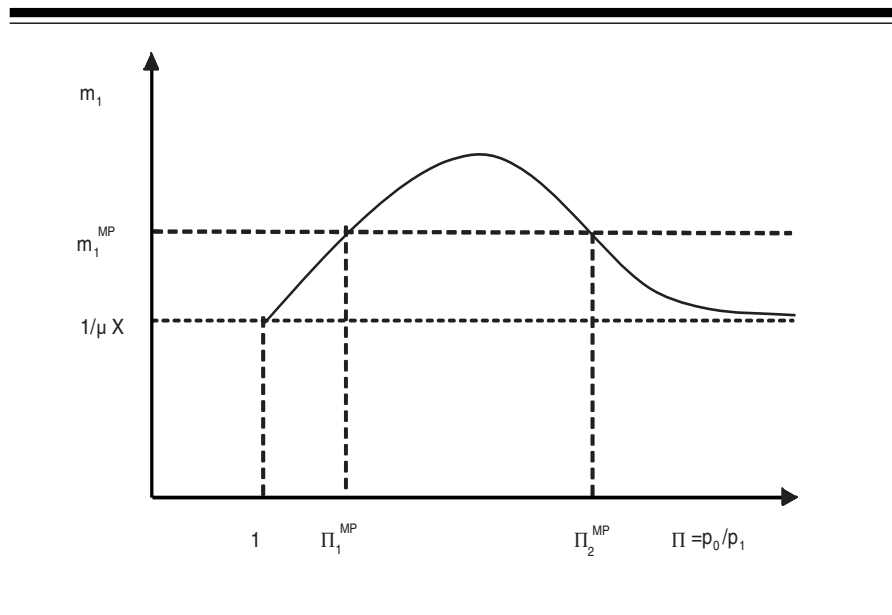


$\pi = 1$, the derivative of allocational efficiency with respect to $p_1$,

$$\partial d / \partial p_1 = \varepsilon p_1^{-\varepsilon-1} \left( \pi^{-\varepsilon-1} - 1 \right), \tag{44}$$

is zero, and we have a contradiction. On the other hand, with positive inflation, the impact of $p_1$ on allocational efficiency is negative. This suggests that the steady state inflation rate is positive, as indeed shown by Wolman (2001). We can find the steady state inflation rate as the solution to the following fix-point problem. Conditional on some inflation rate, $\pi$, use equations (35) and (36) to determine steady state real balances, $m_1$, and marginal cost, $\psi$. Conditional on $(\pi, m_1, \psi)$, use equation (42) to obtain the steady state Lagrange multiplier $\lambda$. Finally, we have to verify that equation (43) is satisfied.

The competitive equilibrium constraint (32), together with the FOCs for optimal policy, (42) and (43), evaluated at their deterministic steady state indeed yield a unique solution for the steady state, $\left( \pi^{MP}, m_1^{MP}, \psi^{MP} \right)$. Note, however, that contingent on the steady state Markov-perfect real balances $m_1^{MP}$, the competitive equilibrium constraint alone is consistent with multiple steady states. In Figure 2, we graph real balances as a function of the inflation rate, $\pi$, based on equation (36). Notice that as the inflation rate increases, real balances first increase and then decline. This means that for a given choice of real balances that is not too high, $m_1 > m_1^{FC} = 1/ (\chi \mu)$, there are two steady state inflation rates.

*Local Properties of the Markov-Perfect Policy*

For a local approximation of the optimal Markov-perfect policy we can show that the policy stabilizes prices around the trend growth path in response to productivity shocks. Because the steady state involves positive inflation, the expressions for the local approximations are quite convoluted, and we do not display them here. Suffice it to say that locally the optimal Markov-perfect solution is of the form

$$\hat{p}_{1t} = \hat{m}_{1t} - \hat{\xi}_t = \hat{\lambda}_t = 0. \tag{45}$$

We can substitute the local approximation of the Markov-perfect policy rule, second and third equalities of (45), into the log-linear approximation of the optimal pricing equation (15) when the steady state has non-zero inflation and get

$$\hat{p}_{1t} = \left[ \beta \frac{(\varepsilon - 1)(1 - \mu \chi m_1) \pi^\varepsilon}{(\varepsilon - 1) \pi^e - \mu \chi m_1 \left( 1 + \varepsilon \pi^{\varepsilon - 1} \right)} \right] \hat{p}_{1,t+1}. \tag{46}$$

Note that for a steady state with zero inflation, the coefficient on the right-hand side term is zero. Since the steady state of the Markov-perfect equilibrium involves only a very small amount of inflation, the coefficient on future prices is close to zero and certainly less than one. Thus, solving the equation forward implies that there exists a unique REE, $\hat{p}_{1t} = 0$.

## 4.  GLOBAL PROPERTIES OF OPTIMAL POLICY

We now show that the policy rule implied by a Markov-perfect optimal policy does not globally implement the optimal policy allocation. We also conjecture that the policy rule implied by the full-commitment policy may not always be implementable. An augmented full-commitment policy rule that can respond to contemporaneous variables as described in Section 2, however, is likely to implement the optimal policy allocation.

For the analysis of the global properties of policy rules, it will be useful to rewrite a firm's profit maximization condition (12), which represents the competitive equilibrium constraint for the planning problem. Solve this expression for a firm's optimal relative price as a markup over the average marginal cost for which the price is set

$$\frac{P_t(j)}{P_t} = \mu \frac{\psi_t + \beta E_t \left[ \psi_{t+1} (P_{t+1}/P_t)^\varepsilon \right]}{1 + \beta E_t \left[ (P_{t+1}/P_t)^{\varepsilon - 1} \right]}. \tag{47}$$

We can think of this expression as a firm's optimal relative price choice on the left-hand side, $p_{0t}$, conditional on the relative prices set by all other firms, $\bar{p}_{0t}$, determining the right-hand side of the equation. The behavior of the other firms is reflected in the equilibrium values of marginal cost and the inflation

rate. For our argument, we will assume that there are no shocks to the economy, that is, productivity is constant. Using the equilibrium conditions (13), (16), and (22) for the right-hand side of (47), we then get

$$p_{0,t} = \mu\chi \frac{m_{1t} p_1 \left(\bar{p}_{0,t}\right) + \beta m_{1,t+1} \bar{p}_{0,t} \pi_{t+1}^{\varepsilon-1}}{1 + \beta\pi_{t+1}^{\varepsilon-1}} \text{ with } \pi_{t+1} = \bar{p}_{0,t}/p_1\left(\bar{p}_{0,t+1}\right).$$
(48)

**Markov-Perfect Policy**

The Markov-perfect policy rule not only stabilizes prices in response to small productivity shocks, but stabilization is the globally optimal response to shocks,

$$m_{1t} = m_1^{MP}\xi_t.$$
(49)

We can verify that (49) is the optimal response to productivity shocks by substituting the expression for $m_{1t}$ into equations (32), (42), and (43). This policy rule reflects the definition of a Markov-perfect policy: it depends only on payoff-relevant state variables, that is, $\xi_t$ only in our case.

In general, the Markov-perfect policy rule cannot implement the planning allocation as a competitive equilibrium outcome. King and Wolman (2004) argue that a Markov-perfect optimal policy introduces strategic complementarities into the firms' price-setting behavior and thereby makes multiple equilibria possible. With constant normalized real balances of the Markov-perfect policy and no productivity shocks, the optimal pricing condition (48) simplifies to

$$p_{0,t} = \mu\chi m_1^{MP} \frac{p_1\left(\bar{p}_{0,t}\right) + \beta\bar{p}_{0,t}\pi_{t+1}^{\varepsilon-1}}{1 + \beta\pi_{t+1}^{\varepsilon-1}}.$$
(50)

Strategic complementarities are said to be present if a representative firm increases its own control variable when it perceives that all other firms increase their control variable. In terms of the price-setting equation (50): a firm increases its own relative price, $p_{0t}$, on the left-hand side of the expression if all other firms increase their relative price, $\bar{p}_{0t}$, on the right-hand side of the expression. Essentially, if all other firms increase their price, $\bar{p}_{0t}$, then the expected inflation rate increases, and therefore a firm will increase its own relative price in order to prevent an erosion of its relative price in the next period. Since the equilibrium relative price is a fix-point of expression (50), $p_{0t} = \bar{p}_{0t}$, strategic complementarities raise the possibility of multiple fixed points, that is, multiple equilibria.

In Figure 3 we graph the RHS of (50), conditional on some value for $p_{1,t+1}$. If we evaluate the RHS of (50) at $\bar{p}_{0t} = 1$, we get $p_{0t} = 1$ and $RHS = \mu\chi m_1^{MP} > 1$. If we consider the limit of the RHS as $\bar{p}_{0t}$ becomes arbitrarily large, we see that $\bar{p}_{1t}$ converges to a finite value and the inflation rate becomes arbitrarily large, thus the RHS converges to a line through the

**Figure 3**
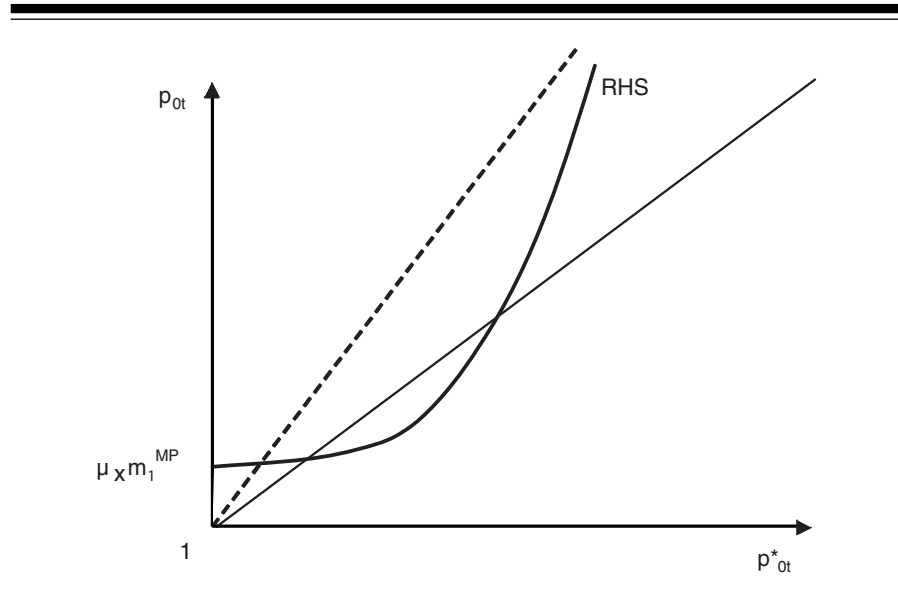


origin with slope $\mu \chi m_1^{MP} > 1$. Without a further analysis of the behavior of the RHS for finite positive values of $\bar{p}_{0t}$, this at least suggests the possibility of two intersection points of the RHS with the diagonal. Furthermore we know that in the steady state, when $p_{1,t+1} = p_1^{MP}$, there are indeed two solutions for $p_0$ to equation (36). King and Wolman (2004) show that, in general, there exist two intersection points. Thus there is no unique equilibrium and the Markov-perfect policy rule does not implement the planning allocation.

### Full-Commitment Policy

Optimal full-commitment monetary policy stabilizes prices in response to productivity shocks not only locally around the steady state, but also globally,

$$m_{1t} = \tilde{m}_{1t}\xi_t, \tilde{m}_1 = \Gamma\left(\lambda_{t-1}\right), p_{1t} = \Theta\left(\lambda_{t-1}\right), \text{ and } \lambda_t = \Omega\left(\lambda_{t-1}\right). \quad (51)$$

To see this, simply note that equations (32), (33), and (34) define a system in $(\tilde{m}_{1t}, p_{1t}, \lambda_{t-1})$ that is independent of productivity shocks. Different from the Markov-perfect policy, the Lagrange multiplier on the competitive equilibrium constraint is not constant and therefore the normalized real balances are not constant.

We do not have unambiguous results on the implementation of the planning allocation through the full-commitment policy rule. On the one hand, we can show that if the Lagrange multiplier has attained its steady state value, $\lambda_{t-1} = \lambda^{FC}$, then the full-commitment policy rule implements the planning

solution. On the other hand, as long as the Lagrange multiplier has not attained its steady state, the full-commitment policy rule suffers from some of the same problems as does the Markov-perfect policy.

Suppose that the Lagrange multiplier has attained its steady state value, $\lambda_{t-1} = \lambda^{FC}$. If we substitute the value for the Lagrange multiplier in the FOCs (33) and (34), we can see that they will always be satisfied from there on. But this means that from there on the normalized real balances attain their steady state value, $m_1^{FC}$, and the competitive equilibrium constraint (32) simplifies to

$$0 = p_{0t}^{1-\varepsilon} \left( 1 - \frac{p_{1t}}{p_{0t}} \right). \tag{52}$$

Therefore $p_{1t} = p_{0t}$, that is $P_t^{FC} = P_{t-1}^{FC}$, and prices are determined.

Now consider the transitional phase when the Lagrange multiplier differs from its steady state value. Given the implied policy rule (51), we can construct future nominal money stocks recursively as functions of the initial value of the Lagrange multiplier

$$M_t = \xi_t \cdot \Gamma\left(\lambda_{t-1}\right) \cdot \Theta\left(\lambda_{t-1}\right) \cdot P_t, \text{ and} \tag{53}$$

$$P_t = \frac{p_{0,t-1}}{p_{1,t}} P_{t-1} = \frac{p_0\left(p_{1,t-1}\right)}{p_{1,t}} P_{t-1} = \frac{p_0\left[\Theta\left(\lambda_{t-2}\right)\right]}{\Theta\left(\lambda_{t-1}\right)} P_{t-1}. \tag{54}$$

With full commitment, a policymaker can always announce a time path for the nominal money supply and follow through on that announcement. Given the nominal money supply rule, we can rewrite the optimal pricing condition (48) in nominal terms and get

$$P_{0t} = \mu\chi \frac{M_t + \beta M_{t+1}\left(P_{t+1}/P_t\right)^{\varepsilon-1}}{1 + \beta\left(P_{t+1}/P_t\right)^{\varepsilon-1}} \tag{55}$$

$$\text{with } \frac{P_{t+1}}{P_t} = \left[ \frac{P_{0,t+1}^{1-\varepsilon} + P_{0,t}^{1-\varepsilon}}{P_{0,t}^{1-\varepsilon} + P_{0,t-1}^{1-\varepsilon}} \right]^{1/(1-\varepsilon)}.$$

As we do for the analysis of the Markov-perfect policy, we are looking for a fix point in the optimal nominal price, $P_{0t}$, conditional on the past and future nominal prices, $P_{0,t-1}$ and $P_{0,t+1}$, and the nominal money stocks, $M_t$ and $M_{t+1}$. Clearly for a constant money supply, that is, the constant steady state Lagrange multiplier, there is a unique solution for $P_{0t}$. If the Lagrange multiplier converges globally to its steady state, then if the difference between $M_t$ and $M_{t+1}$ is small enough, we will also have a unique solution. We do not, however, prove that there is a unique solution for the initial phase of the transition period.

Note that for full-commitment policy, we have only outlined the same potential for multiple equilibria as King and Wolman (2004) have shown to exist for the Markov-perfect policy rule. We have not proven that the full-commitment policy rule cannot implement the planning allocation. Whether or not the full-commitment policy rule implements the planning allocation

may be irrelevant if one believes that a policymaker can always respond to contemporaneous variables. If such a response is feasible, then an augmented full-commitment policy rule as described in Section 2 may always implement the planning allocation.

## 5.   CONCLUSION

This paper has considered optimal monetary policy as the solution to both full-commitment and time-consistent Markov-perfect planning problems. The solutions are consistent with rational expectations competitive equilibria. The optimal solution to the planning problem implies a rule for the assumed policy instrument, in our case, a money supply instrument. We have then verified that, for local approximations to the solution of the optimal policy problem, the implied policy rules implement the planning allocations, that is, the planning allocation is the unique rational expectations equilibrium conditional on the implied policy rule. However, following on the insights of King and Wolman (2004), we have then examined whether the implied policy rules also implement the allocation globally. We find that a money supply rule that is Markov-perfect does not implement the planning solution. We provide a partial argument that the full-commitment money supply rule does implement the planning solution, but we do not have a complete proof for this statement.

For the analysis, we have taken the choice of monetary instrument, in this case the nominal money stock, as given but this choice is not innocuous. In other work (Dotsey and Hornstein 2005), we have argued that equilibrium indeterminacy may depend on the choice of policy instrument. In particular, if the Markov-perfect policy uses the nominal interest rate as an instrument, the equilibrium is determinate.

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