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Non-Stationarity and Instability in Small Open-Economy Models Even When They Are "Closed"

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Open economies are characterized by the ability to trade goods both intra- and intertemporally, that is, their residents can move goods and assets across borders and over time. These transactions are reflected in the current account, which measures the value of a country's export and imports, and its mirror image, the capital account, which captures the accompanying exchange of assets. The current account serves as a shock absorber, which agents use to optimally smooth their consumption. The means for doing so are borrowing and lending in international financial markets. It almost goes without saying that international macroeconomists have had a long-standing interest in analyzing the behavior of the current account.

The standard intertemporal model of the current account conceives a small open economy as populated by a representative agent who is subject to fluctuations in his income. By having access to international financial markets, the agent can lend surplus funds or make up shortfalls for what is necessary to maintain a stable consumption path in the face of uncertainty. The international macroeconomics literature distinguishes between an international asset market that is incomplete and one that is complete. The latter describes a modeling framework in which agents have access to a complete set of state-contingent securities (and, therefore, can share risk perfectly); when markets

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are incomplete, on the other hand, agents can only trade in a restricted set of assets, for instance, a bond that pays fixed interest.

The small open-economy model with incomplete international asset markets is the main workhorse in international macroeconomics. However, the baseline model has various implications that may put into question its usefulness in studying international macroeconomic issues. When agents decide on their intertemporal consumption path they trade off the utility-weighted return on future consumption, measured by the riskless rate of interest, against the return on present consumption, captured by the time discount factor. The basic set-up implies that expected consumption growth is stable only if the two returns exactly offset each other, that is, if the product of the discount factor and the interest rate equal one. The entire optimization problem is ill-defined for arbitrary interest rates and discount factors as consumption would either permanently decrease or increase.[1]

Given this restriction on two principally exogenous parameters, the model then implies that consumption exhibits random-walk behavior since the effects of shocks to income are buffered by the current account to keep consumption smooth. The random-walk in consumption, which is reminiscent of Hall's (1978) permanent income model with linear-quadratic preferences, is problematic because it implies that all other endogenous variables inherit this nonstationarity so that the economy drifts over time arbitrarily far away from its initial condition. To summarize, the standard small open-economy model with incomplete international asset markets suffers from what may be labelled the unit-root problem. This raises several issues, not the least of which is the overall validity of the solution in the first place, and its usefulness in conducting business cycle analysis.

In order to avoid this unit-root problem, several solutions have been suggested in the literature. Schmitt-Grohé and Uribe (2003) present an overview of various approaches. In this article, I am mainly interested in inducing stationarity by assuming a debt-elastic interest rate. Since this alters the effective interest rate that the economy pays on foreign borrowing, the unit root in the standard linearized system is reduced incrementally below unity. This preserves a high degree of persistence, but avoids the strict unit-root problem. Moreover, a debt-elastic interest rate has an intuitive interpretation as an endogenous risk premium. It implies, however, an additional, essentially ad hoc feedback mechanism between two endogenous variables. Similar to the literature on the determinacy properties of monetary policy rules or models with

[1] Conceptually, the standard current account model has a lot of similarities to a model of intertemporal consumer choice with a single riskless asset. The literature on the latter gets around some of the problems detailed here by, for instance, imposing borrowing constraints. Much of that literature is, however, mired in computational complexities as standard linearization-based solution techniques are no longer applicable.

increasing returns to scale, the equilibrium could be indeterminate or even non-existent.

I show in this article that commonly used specifications of the risk premium do not lead to equilibrium determinacy problems. In all specifications, indeterminacy of the rational expectations equilibrium can be ruled out, although in some cases there can be multiple steady states. It is only under a specific assumption on whether agents internalize the dependence of the interest rate on the net foreign asset position that no equilibrium may exist.

I proceed by deriving, in the next section, an analytical solution for the (linearized) canonical small open-economy model which tries to illuminate the extent of the unit-root problem. Section 2 then studies the determinacy properties of the model when a stationarity-inducing risk-premium is introduced. In Section 3, I investigate the robustness of the results by considering different specifications that have been suggested in the literature. Section 4 presents an alternative solution to the unit-root problem via portfolio adjustment costs, while Section 5 summarizes and concludes.

1. THE CANONICAL SMALL OPEN-ECONOMY MODEL

Consider a small open economy that is populated by a representative agent[2] whose preferences are described by the following utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right), \tag{1}$$

where $0 < \beta < 1$ and E_t is the expectations operator conditional on the information set at time t. The period utility function u obeys the usual Inada conditions which guarantee strictly positive consumption sequences $\{c_t\}_{t=0}^{\infty}$. The economy's budget constraint is

$$c_t + b_t \leq y_t + R_{t-1} b_{t-1}, \tag{2}$$

where y_t is stochastic endowment income; R_t is the gross interest rate at which the agent can borrow and lend b_t on the international asset market. The initial condition is $b_{-1} \gtreqless 0$. In the canonical model, the interest rate is taken parametrically.

The agent chooses consumption and net foreign asset sequences $\{c_t, b_t\}_{t=0}^{\infty}$ to maximize (1) subject to (2). The usual transversality condition applies. First-order necessary conditions are given by

$$u'\left(c_t\right) = \beta R_t E_t u'\left(c_{t+1}\right), \tag{3}$$

[2] In what follows, I use the terms "agent," "economy," and "country," interchangeably. This is common practice in the international macro literature and reflects the similarity between small open-economy models and partial equilibrium models of consumer choice.

and the budget constraint (2) at equality. The Euler equation is standard. At the margin, the agent is willing to give up one unit of consumption, valued by its marginal utility, if he is compensated by an additional unit of consumption next period augmented by a certain (properly discounted) interest rate, and evaluated by its uncertain contribution to utility. Access to the international asset market thus allows the economy to smooth consumption in the face of uncertain domestic income. Since the economy can only trade in a single asset such a scenario is often referred to as one of "incomplete markets." This stands in contrast to a model where agents can trade a complete set of state-contingent assets ("complete markets").

In what follows, I assume for ease of exposition that y_t is $i.i.d.$ with mean $\overline{y}$, and that the interest rate is constant and equal to the world interest rate $R^* > 1$. The latter assumption will be modified in the next section. Given these assumptions a steady state only exists if $\beta R^* = 1$. Steady-state consumption is, therefore, $\overline{c} = \overline{y} + \frac{1-\beta}{\beta}\overline{b}$. Since consumption is strictly positive, this imposes a restriction on the admissible level of net foreign assets $\overline{b} > -\frac{\beta}{1-\beta}\overline{y}$. The structure of this model is such that it imposes a restriction on the two principally structural parameters β and R^*, which is theoretically and empirically problematic; there is no guarantee or mechanism in the model that enforces this steady-state restriction to hold. Even more so, the steady-state level of a choice variable, namely net foreign assets $\overline{b}$, is not pinned down by the model's optimality conditions. Instead, there exists a multiplicity of steady states indexed by the initial condition $\overline{b} = b_{-1}$.[3]

Despite these issues, I now proceed by linearizing the first-order conditions around the steady state for some $\overline{b}$. Denoting $\widetilde{x}_t = \log x_t - \log \overline{x}$ and $\widehat{x}_t = x_t - \overline{x}$, the linearized system is[4]

$$
\begin{aligned}
E_t \widetilde{c}_{t+1} &= \widetilde{c}_t, & (4)\\
\overline{c}\widetilde{c}_t + \widehat{b}_t &= \overline{y}\widetilde{y}_t + \beta^{-1}\widehat{b}_{t-1}. & (5)
\end{aligned}
$$

It can be easily verified that the eigenvalues of this dynamic system in $\left[\widetilde{c}_t, \widehat{b}_t\right]$ are $\lambda_1 = 1, \lambda_2 = \beta^{-1} > 1$. Since $\widehat{b}$ is a pre-determined variable this results in a unique rational expectations equilibrium for all admissible parameter values. The dynamics of the solution are given by (a detailed derivation of the solution

[3] In the international real business cycle literature, for instance, Baxter and Crucini (1995), $\overline{b}$ is, therefore, often treated as a parameter to be calibrated.

[4] Since the interest rate is constant, the curvature of the utility function does not affect the time path of consumption and, consequently, does not appear in the linearization. Moreover, net foreign assets are approximated in levels since b_t can take on negative values or zero, for which the logarithm is not defined.

can be found in the Appendix)

$$\widetilde{c}_t = \frac{1-\beta}{\beta}\frac{\widehat{b}_{t-1}}{\overline{c}} + (1-\beta)\frac{\overline{y}}{\overline{c}}\widetilde{y}_t, \tag{6}$$

$$\frac{\widehat{b}_t}{\overline{c}} = \frac{\widehat{b}_{t-1}}{\overline{c}} + \beta\frac{\overline{y}}{\overline{c}}\widetilde{y}_t. \tag{7}$$

The contemporaneous effect of a 1 percent innovation to output is to raise foreign lending as a fraction of steady-state consumption by $\beta\frac{\bar{y}}{\bar{c}}$ percent, which is slightly less than unity in the baseline case $\overline{b} = 0$. In line with the permanent income hypothesis only a small percentage of the increase in income is consumed presently, so that future consumption can be raised permanently by $\frac{1-\beta}{\beta}$. The non-stationarity of this solution, the "unit-root problem," is evident from the unit coefficient on lagged net foreign assets in (7). Temporary innovations have, therefore, permanent effects; the endogenous variables wander arbitrarily far from their starting values. This also means that the unconditional second moments, which are often used in business cycle analysis to evaluate a model, do not exist.

Moreover, the solution is based on an approximation that is technically only valid in a small neighborhood around the steady state. This condition will be violated eventually with probability one, thus ruling out the validity of the linearization approach in the first place. Since an equation system such as (4)–(5) is at the core of much richer open-economy models, the non-stationarity of the incomplete markets solution carries over. The unit-root problem thus raises the question whether (linearized) incomplete market models offer accurate descriptions of open economies. In the next sections, I study the equilibrium properties of various modifications to the canonical model that have been used in the literature to "fix" the unit-root problem.[5]

2. INDUCING STATIONARITY VIA A DEBT-ELASTIC INTEREST RATE

The unit-root problem arises because of the random-walk property of consumption in the linearized Euler equation (4). Following Schmitt-Grohé and Uribe (2003) and Senhadji (2003), a convenient solution is to make the interest rate the economy faces a function of net foreign assets $R_t = F\left(b_t - \overline{b}\right)$,

[5] In most of the early international macro literature, the unit-root problem tended to be ignored despite, in principle valid, technical problems. The unit root is transferred to the variables of interest, such as consumption, on the order of the net interest rate, which is quantitatively very small (in the present example, $\frac{1-\beta}{\beta}$). While second moments do not exist in such a non-stationary environment, researchers can still compute sample moments to perform business cycle analysis. Moreover, Schmitt-Grohé and Uribe (2003) demonstrate that the dynamics of the standard model with and without the random walk in endogenous variables are quantitatively indistinguishable over a typical time horizon. Their article, thus, gives support for the notion of using the incomplete market setup for analytical convenience.

where F is decreasing in b, $\overline{b}$ is the steady-state value of b, and $F(0) = R^*$. If a country is a net foreign borrower, it pays an interest rate that is higher than the world interest rate. The reference point for the assessment of the risk premium is the country's steady state. Intuitively, $\overline{b}$ represents the level of net foreign assets that is sustainable in the long run, either by increasing (if positive) or decreasing (if negative) steady-state consumption relative to the endowment.

If a country deviates in its borrowing temporarily from what international financial markets perceive as sustainable in the long run, it is penalized by having to pay a higher interest rate than "safer" borrowers. This has the intuitively appealing implication that the difference between the world interest rate and the domestically relevant rate can be interpreted as a risk premium. The presence of a debt-elastic interest rate can be supported by empirical evidence on the behavior of spreads, that is, the difference between a country's interest rate and a benchmark rate, paid on sovereign bonds in emerging markets (Neumeyer and Perri, 2005). Relative to interest rates on U.S. Treasuries, the distribution of spreads has a positive mean, and they are much more volatile.

A potential added benefit of using a debt-elastic interest rate is that proper specification of F may allow one to derive the steady-state value of net foreign assets endogenously. However, the introduction of a new, somewhat arbitrary link between endogenous variables raises the possibility of equilibrium indeterminacy and non-existence similar to what is found in the literature on monetary policy rules and production externalities. I study two cases. In the first case, the small open economy takes the endogenous interest rate as given. That is, the dependence of the interest rate on the level of outstanding net assets is not internalized. The second case assumes that agents take the feedback from assets to interest rates into account.

No Internalization

The optimization problem for the small open economy is identical to the canonical case discussed above. The agent does not take into account that the interest charged for international borrowing depends on the amount borrowed. Analytically, the agent takes R_t as given. The first-order conditions are consequently (2) and (3). Imposing the interest rate function $R_t = F\left(b_t - \overline{b}\right)$ yields the Euler equation when the risk premium is not internalized:

$$u'(c_t) = \beta F\left(b_t - \overline{b}\right) E_t u'(c_{t+1}). \tag{8}$$

The Euler equation highlights the difference to the canonical model. Expected consumption growth now depends on an endogenous variable, which tilts the consumption path away from random-walk behavior. However, existence of a steady state still requires $R = R^* = \beta^{-1}$. Despite the assumption of an endogenous risk premium, this model suffers from the same deficiency as the canonical model in that the first-order conditions do not fully pin down all endogenous variables in steady state.[6]

After substituting the interest rate function, the first-order conditions are linearized around some steady state $\overline{b}$. I impose additional structure by assuming that the period utility function $u(c) = \frac{c^{1-1/\sigma}-1}{1-1/\sigma}$, where $\frac{u''(c)c}{u'(c)} = -1/\sigma$, and $\sigma > 0$ is the intertemporal substitution elasticity. Since I am mainly interested in the determinacy properties of the model, I also abstract from time variation in the endowment process $y_t = y, \forall t$. Furthermore, I assume that $F'(0) = -\psi$.[7] The linearized equation system is then

$$\begin{aligned} E_t \widetilde{c}_{t+1} &= \widetilde{c}_t - \beta\sigma\psi\widehat{b}_t, \\ \overline{c}\widetilde{c}_t + \widehat{b}_t &= \left(\frac{1}{\beta} - \psi\overline{b}\right)\widehat{b}_{t-1}. \end{aligned} \tag{9}$$

The reduced-form coefficient matrix of this system can be obtained after a few steps:

$$\begin{bmatrix} 1 & -\beta\sigma\psi \\ -\overline{c} & \frac{1}{\beta} + \left(\beta\sigma\overline{c} - \overline{b}\right)\psi \end{bmatrix}, \tag{10}$$

where $\overline{c} = y + \frac{1-\beta}{\beta}\overline{b}$ as before. I can now establish

Proposition 1 *In the model with additively separable risk premium and no internalization, there is a unique equilibrium for all admissible parameter values.*

Proof. In order to investigate the determinacy properties of this model, I first compute the trace $tr = 1 + \frac{1}{\beta} + \left(\beta\sigma\overline{c} - \overline{b}\right)\psi$ and the determinant $det = \frac{1}{\beta} - \psi\overline{b}$. Since there is one predetermined variable, a unique equilibrium requires one root inside and one root outside the unit circle. Both (zero) roots inside the unit circle imply indeterminacy (non-existence). The Appendix shows that determinacy requires $|tr| > 1 + det$, while $|det| \lesseqgtr 1$. The first condition reduces to $\beta\sigma\psi\overline{c} > 0$, which is always true because of strictly positive consumption. Note also that $tr > 1 + det$. Indeterminacy and non-existence require $|tr| < 1 + det$, which cannot hold because of positive consumption. The proposition then follows immediately.

[6] This is an artifact of the assumption of no internalization and the specific assumptions on the interest rate function.

[7] An example of a specific functional form that is consistent with these assumptions and that has been used in the literature (e.g., Schmitt-Grohé and Uribe 2003) is

$$R_t = R^* + \psi\left[e^{-\left(b_t - \overline{b}\right)} - 1\right].$$

Internalization

An alternative scenario assumes that the agent explicitly takes into account that the interest rate he pays on foreign borrowing depends on the amount borrowed. Higher borrowing entails higher future debt service which reduces the desire to borrow. The agent internalizes the cost associated with becoming active on the international asset markets in that he discounts future interest outlays not at the world interest rate but at the domestic interest rate, which is inclusive of the risk premium.[8]

The previous assumptions regarding the interest rate function and the exogenous shock remain unchanged. Since the economy internalizes the dependence of the interest rate on net foreign assets, the first-order conditions change. Analytically, I substitute the interest rate function into the budget constraint (2) before taking derivatives, thereby eliminating R from the optimization problem. The modified Euler equation is

$$u'(c_t) = \beta F\left(b_t - \overline{b}\right)[1 + \varepsilon_F(b_t)]\, E_t u'(c_{t+1}), \tag{11}$$

where $\varepsilon_F(b_t) = \frac{F'(b_t - \overline{b})b_t}{F(b_t - \overline{b})}$ is the elasticity of the interest rate function with respect to net foreign assets. Compared to the case of no internalization, the effective interest rate now includes an additional term in the level of net foreign assets. Whether the steady-state level of $\overline{b}$ is determined, therefore, depends on this elasticity. Maintaining the assumption $F'(0) = -\psi$, it follows that $\varepsilon_F(\overline{b}) = -\psi R^* \overline{b}$.

This provides the additional restriction needed to pin down the steady state:

$$\overline{b} = \frac{R^* - 1/\beta}{\psi}. \tag{12}$$

If the country's discount factor is bigger than $1/R^*$, that is, if it is more patient than those in the rest of the world, its steady-state asset position is strictly positive. A more impatient country, however, accumulates foreign debt to finance consumption. Note further that $R = R^*$, but not necessarily equals β^{-1}, while $\overline{b}$ asymptotically reaches zero as ψ grows large. It is worth emphasizing that $\beta R^* = 1$ is no longer a necessary condition for the existence of a steady state, and that $\overline{b}$ is, in fact, uniquely determined. Internalization of the risk premium, therefore, avoids one of the pitfalls of the standard model, but it also nicely captures the idea that some countries appear to have persistent levels of foreign indebtedness.

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[8] The difference between internalization and no internalization of the endogenous risk premium is also stressed by Nason and Rogers (2006). Strictly speaking, with internalization the country stops being a price-taker in international asset markets. This is analogous to open-economy models of "semi-small" countries that are monopolistically competitive and price-setting producers of export goods. Schmitt-Grohé (1997) has shown that feedback mechanisms of this kind are important sources of non-determinacy of equilibria.

I now proceed by linearizing the equation system:

$$\begin{aligned} E_t\widetilde{c}_{t+1} &= \widetilde{c}_t - \beta\sigma\psi\left(2-\overline{b}\right)\widehat{b}_t, \\ \overline{c}\widetilde{c}_t + \widehat{b}_t &= \left(R^* - \psi\overline{b}\right)\widehat{b}_{t-1}. \end{aligned} \tag{13}$$

The coefficient matrix that determines the dynamics can be derived as:

$$\begin{bmatrix} 1 & -\beta\sigma\psi(2-\overline{b}) \\ -\bar{c} & \frac{1}{\beta} + \beta\sigma\psi(2-\overline{b})\overline{c} \end{bmatrix}, \tag{14}$$

where now $\overline{b} = \frac{R^*-1/\beta}{\psi}$ and $\overline{c} = \overline{y} + (R^*-1)\,\overline{b}$. The determinacy properties of this case are given in

Proposition 2 *In the model with additively separable risk premium and internalization, the equilibrium is unique if and only if*

$$\overline{b} < 2,$$

or

$$\overline{b} > 2 + 2\frac{1+\beta}{\beta}\frac{1}{\beta\sigma\psi\overline{c}}.$$

No equilibrium exists otherwise.

Proof. The determinant of the system matrix is $det = \beta^{-1} > 1$. This implies that there is at least one explosive root, which rules out indeterminacy. Since the system contains one jump and one predetermined variable, a unique equilibrium requires $|tr| > 1 + det$, where $tr = 1 + \beta^{-1} + \beta\sigma\psi\overline{c}(2-\overline{b})$. The lower bound of the condition establishes that $\beta\sigma\psi(2-\overline{b})\overline{c} > 0$. Since $\overline{c} > 0$, it must be that $\overline{b} < 2$. From $-tr > 1 + det$, the second part of the determinacy region follows after simply rearranging terms. The proposition then follows immediately.

The proposition shows that a sufficient condition for determinacy is that the country is a net foreign borrower, which implies $\beta^{-1} > R^*$. A relatively impatient country borrows from abroad to sustain current consumption. Since this incurs a premium above the world interest rate, the growth rate of debt is below that of, say, the canonical case, and debt accumulation is, therefore, nonexplosive. Even if the country is a net foreign lender, determinacy can still be obtained for $0 < \overline{b} < 2$ or $R^* < \beta^{-1} + 2\psi$. A slightly more patient country than the rest of the world would imply a determinate equilibrium if the (internalized) interest rate premium is large enough.

From a technical point of view, non-existence arises if both roots in (13) are larger than unity, so that both difference equations are unstable. The budget constraint then implies an explosive time path for assets b which would violate transversality. This is driven by explosive consumption growth financed by interest receipts on foreign asset holdings. In the non-existence region, these are large so as not to be balanced by the decline in the interest rate. Effectively, the economy both over-consumes and over-accumulates assets, which cannot be an equilibrium. The only possible equilibrium is, therefore, at the (unique) steady state, while dynamics around it are explosive. This highlights the importance of the elasticity term $1 + \varepsilon_F(b_t)$ in equation (11), which has the power to tilt the consumption away from unit-root (and explosive) behavior for the right parameterization.

As the proposition shows, the non-existence region has an upper bound beyond which the equilibrium is determinate again. The following numerical example using baseline parameter values[9] demonstrates, however, that this boundary is far above empirically reasonable values. Figure 1, Panels A and B depict the determinacy regions for net foreign assets for varying values of σ and ψ. Note that below the lower bound $\overline{b} = 2$ the equilibrium is always determinate, while the size of the non-existence region is decreasing in the two parameters. Recall from equation (12) that the steady-state level $\overline{b}$ depends on the spread between the world interest rate and the inverse of the discount factor. Non-existence, therefore, arises if $\psi < \frac{1}{2}\left(R^* - \beta^{-1}\right)$. In other words, if there is a large wedge between R^* and β^{-1}, a researcher has to be careful not to choose an elasticity parameter ψ that is too small.

Normalizing output $\overline{y} = 1$, the boundary lies at an asset level that is twice as large as the country's GDP. While this is not implausible, net foreign asset holdings of that size are rarely observed. However, choosing a different normalization, for instance, $\overline{y} = 10$ presents a different picture, in which a plausible calibration for, say, a primary resource exporter, renders the solution of the model non-existent. On the other hand, as $\overline{y}$ becomes large, the upper bound for the non-existence region in Figure 1, Panels A and B moves inward, thereby reducing its size. The conclusion for researchers interested in studying models of this type is to calibrate carefully. Target levels for the net-foreign asset to GDP ratio cannot be chosen independently of the stationarity-inducing parameter ψ if equilibrium existence problems are to be avoided. It is worth pointing out again that indeterminacy, and thus the possibility of sunspot equilibria, can be ruled out in this model.

While it is convenient to represent the boundaries of the determinacy region for net foreign assets b, it is nevertheless an endogenous variable, as is c. The parameter restriction in the above proposition can be rearranged in terms of R^*. That is, the economy has a unique equilibrium if either $R^* < \beta^{-1} + 2\psi$ or $R^* > \beta^{-1} + 2\psi\left[1 + \frac{1+\beta}{\beta}\frac{1}{\beta\sigma\left\{\psi y + (R^*-1)\left(R^* - \beta^{-1}\right)\right\}}\right]$. Again, the equilibrium is non-existent otherwise. Since the second term in brackets is strictly positive, the region of non-existence is nonempty. Although the upper bound is still a function of R^* (and has to be computed numerically), this version presents more intuition.

[9] Parameter values used are $\beta = 0.98$, $\sigma = 1$, $\psi = 0.001$, and $\overline{y} = 1$.

Figure 1 Determinacy Regions for Net Foreign Assets *b* and Interest Rates *R**

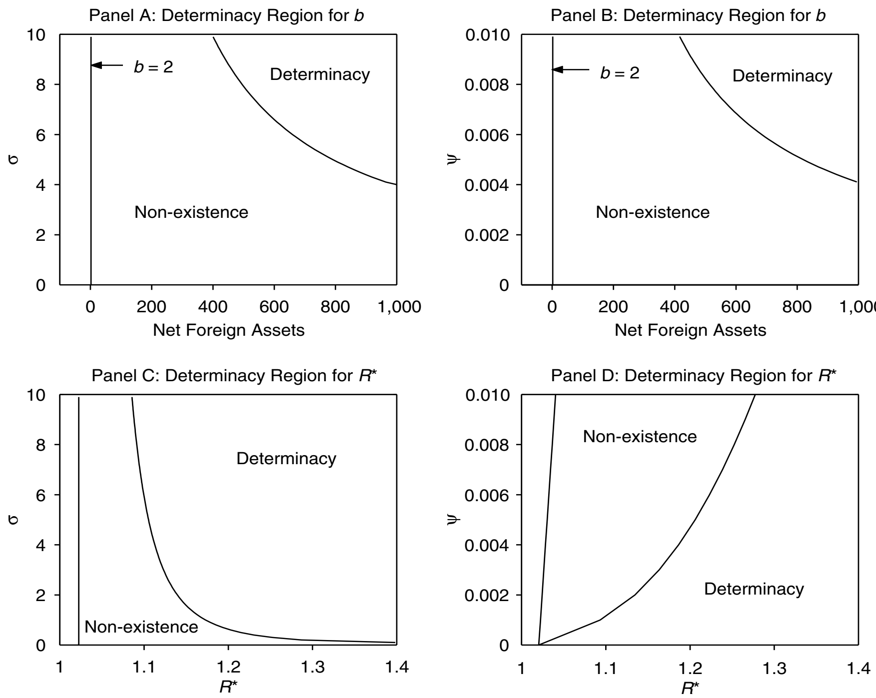

Figure 1, Panels C and D depict the determinacy regions for R^* with varying σ and ψ, respectively. The lower bound of the non-existence region is independent of σ, but increasing in ψ. For a small substitution elasticity, the equilibrium is non-existent unless the economy is more impatient than the rest of the world, inclusive of a factor reflecting the risk premium. This is both consistent with a negative steady-state asset position as well as a small, positive one as long as $b < 2$. Figure 1, Panel D shows that no equilibrium exists even for very small values of ψ. If the economy is a substantial net saver, then the equilibrium is determinate if the world interest rate is (implausibly) high. Analytically, this implies that the asset accumulation equation remains explosive even though there is a large premium to be paid.

To summarize, introducing a debt-elastic interest rate addresses two issues arising in incomplete market models of open economies, viz., the indeterminacy of the steady-state allocation and the induced non-stationarity of the linearized solution. If the derivative of the interest rate function with respect to net asset holdings is nonzero, then the linearized solution is stationary. In the special case when the economy internalizes the dependence of the interest rate on net foreign assets, the rational expectations equilibrium can be nonexistent. However, this situation only arises for arguably extreme parameter values. A nonzero elasticity of the interest rate function is also necessary for the determinacy of the steady state. It is not sufficient, however, as the special case without internalization demonstrated.

3. ALTERNATIVE SPECIFICATIONS

The exposition above used the general functional form $R_t = F\left(b_t - \overline{b}\right)$, with $F(0) = R^*$ and $F'(0) = -\psi$. A parametric example for this function would be additive in the risk premium term, i.e., $R_t = R^* + \psi\left[e^{-\left(b_t - \overline{b}\right)} - 1\right]$. Alternatively, the risk premium could also be chosen multiplicatively, $R_t = R^*\psi(b_t)$, with $\psi(b) = 1$, $\psi' < 0$. With internalization, the Euler equation can then be written as:

$$u'(c_t) = \beta R^* \psi(b_t)\left[1 + \varepsilon_F(b_t)\right] E_t u'(c_{t+1}). \tag{15}$$

$\varepsilon_F(b_t)$ is the elasticity of the risk premium function with respect to foreign assets. Again, the first-order condition shows how a debt-elastic interest rate tilts consumption away from pure random-walk behavior.

A specific example for the multiplicative form of the interest rate function is $R_t = R^* e^{-\psi\left(b_t - \overline{b}\right)}$, which in log-linear form conveniently reduces to $\widetilde{R}_t = -\psi\widehat{b}_t$. Assuming no internalization, the steady state is again not pinned down so that $R = R^* = \beta^{-1}$, and the above restrictions on $\overline{b}$ apply. Internalization of the risk premium leads to $\overline{b} = \frac{R^* - 1/\beta}{\psi R^*}$. Again, the economy is a net saver when it is more patient than the rest of the world. As opposed to the case of an additive premium, the equilibrium is determinate for the entire parameter space. This can easily be established in

Proposition 3 *In the model with multiplicative risk premium, with either internalization or no internalization, the equilibrium is unique for all parameter values.*

Proof. See Appendix.

Nason and Rogers (2006) suggest a specification for the risk premium that is additive in net foreign assets relative to aggregate income: $R_t = R^* - \psi\frac{b_t}{y_t}$.[10] The difference to the additive premium considered above is that even without internalization, foreign and domestic rates need not be the same in the steady state. In the latter case, $\overline{b} = \frac{R^* - 1/\beta}{\psi}$, whereas with internalization, $\overline{b} = \frac{1}{2}\frac{R^* - 1/\beta}{\psi}$. This shows that the endogenous risk premium reduces asset

[10] Note that in this case the general form specification of the interest rate function is $R_t = F(b_t)$, and not $R_t = F(b_t - \overline{b})$.

accumulation when agents take into account the feedback effect on the interest rate. The determinacy properties of this specification are established in

Proposition 4 *If the domestic interest rate is given by $R_t = R^* - \psi \frac{b_t}{y_t}$, under either internalization or no internalization, the equilibrium is unique for all parameter values.*

Proof. See Appendix.

It may appear that the determinacy properties are pure artifacts of the linearization procedure. While I *log*-linearized consumption, functions of b_t were approximated in levels as net foreign assets may very well be negative or zero.[11] Dotsey and Mao (1992), for instance, have shown that the accuracy of linear approximation procedures depends on the type of linearization chosen. It can be verified,[12] however, that this is not a problem in this simple model as far as the determinacy properties are concerned. The coefficient matrix for all model specifications considered is invariant to the linearization.

4. PORTFOLIO ADJUSTMENT COSTS

Finally, I consider one approach to the unit-root problem that does not rely on feedback from net foreign assets to the interest rate. Several authors, for example, Schmitt-Grohé and Uribe (2003) and Neumeyer and Perri (2005), have introduced quadratic portfolio adjustment costs to guarantee stationarity. It is assumed that agents have to pay a fee in terms of lost output if their transactions on the international asset market lead to deviations from some long-run (steady-state) level $\overline{b}$. The budget constraint is thus modified as follows:

$$c_t + b_t + \frac{\psi}{2}\left(b_t - \overline{b}\right)^2 = y_t + R^* b_{t-1}, \tag{16}$$

where $\psi > 0$, and the interest rate on foreign assets is equal to the constant world interest rate R^*. The Euler equation is

$$u'(c_t)\left[1 + \psi\left(b_t - \overline{b}\right)\right] = \beta R^* E_t u'(c_{t+1}). \tag{17}$$

If the economy wants to purchase an additional unit of foreign assets, current consumption declines by one plus the transaction cost $\psi\left(b_t - \overline{b}\right)$. The payoff for the next period is higher consumption by one unit plus the fixed (net) world interest rate.

Introducing this type of portfolio adjustment costs does not pin down the steady-state value of $\overline{b}$. The Euler equation implies the same steady-state restriction as the canonical model, namely $\beta R^* = 1$ and $\overline{b} > -\frac{\beta}{1-\beta}\overline{y}$.

[11] The interpretation of the linearized system in terms of percentage deviations from the steady state can still be preserved by expressing foreign assets relative to aggregate income or consumption, as in equation (7).

[12] Details are available from the author upon request.

However, the Euler equation (17) demonstrates the near equivalence between the debt-dependent interest rate function and the debt-dependent-borrowing cost formulation. The key to avoiding a unit root in the dynamic model is to generate feedback that tilts expected consumption growth, which can be achieved in various ways.

The coefficient matrix of the two-variable system in $\left[\widetilde{c}_t, \widehat{b}_t\right]$ is given by

$$\begin{bmatrix} 1 & -\sigma\psi \\ -\overline{c} & \beta^{-1}+\sigma\psi\overline{c} \end{bmatrix}.$$

It can be easily verified that both eigenvalues are real and lie on opposite sides of the unit circle over the entire admissible parameter space. The rational expectations solution is, therefore, unique. The same conclusion applies when different linearization schemes, as previously discussed, are used.

It is worthwhile to point out that Schmitt-Grohé and Uribe (2003) have suggested that the model with portfolio adjustment costs and the model with a debt-elastic interest rate imply similar dynamics. Inspection of the two respective Euler equations reveals that the debt-dependent discount factors in the linearized versions are identical for a properly chosen parameterization. However, portfolio costs do not appear in the linearized budget constraint, since they are of second order, whereas the time-varying interest rate changes debt dynamics in a potentially critical way. It follows, that this assertion is true only for that part of the parameter space that results in a unique solution, but a general equivalence result, such as between internalized and external risk premia, cannot be derived.

5. CONCLUSION

Incomplete market models of small open economies imply non-stationary equilibrium dynamics. Researchers who want to work with this type of model are faced with a choice between theoretical rigor and analytical expediency in terms of a model solution. In order to alleviate this tension, several techniques to induce stationarity have been suggested in the literature. This article has investigated the determinacy properties of models with debt-elastic interest rates and portfolio adjustment costs. The message is a mildly cautionary one. Although analytically convenient, endogenizing the interest rate allows for the possibility that the rational expectations equilibrium does not exist. I show that an additively separable risk premium with a specific functional form that is used in the literature can imply non-existence for a plausible parameterization. I suggest alternative specifications that are not subject to this problem. In general, however, this article shows that the determinacy properties depend on specific functional forms, which is not readily apparent a priori.

A question that remains is to what extent the findings in this article are relevant in richer models. Since analytical results may not be easily available, this remains an issue for further research. Moreover, there are other suggested solutions to the unit-root problem. As the article has emphasized, the key is to tilt expected consumption growth away from unity. I have only analyzed approaches that work on endogenizing the interest rate, but just as conceivably the discount factor β could depend on other endogenous variables as in the case of Epstein-Zin preferences. The rate at which agents discount future consumption streams might depend on their utility level, which in turn depends on consumption and net foreign assets. Again, this would provide a feedback mechanism from assets to the consumption tilt factor. Little is known about equilibrium determinacy properties under this approach.

APPENDIX

Solving the Canonical Model

The linearized equation system describing the dynamics of the model is

$$\begin{aligned} E_t \widetilde{c}_{t+1} &= \widetilde{c}_t, \\ \overline{c}\widetilde{c}_t + \widehat{b}_t &= \overline{y}\widetilde{y}_t + \beta^{-1}\widehat{b}_{t-1}. \end{aligned}$$

I solve the model by applying the method described in Sims (2002). In order to map the system into Sims's framework, I define the endogenous forecast error η_t as follows:

$$\widetilde{c}_t = \xi^c_{t-1} + \eta_t = E_{t-1}\widetilde{c}_t + \eta_t.$$

The system can then be rewritten as:

$$\begin{bmatrix} 1 & 0 \\ \overline{c} & 1 \end{bmatrix} \begin{bmatrix} \xi^c_t \\ \widehat{b}_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \beta^{-1} \end{bmatrix} \begin{bmatrix} \xi^c_{t-1} \\ \widehat{b}_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \overline{y} \end{bmatrix} \widetilde{y}_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \eta_t.$$

Invert the lead matrix $\begin{bmatrix} 1 & 0 \\ \overline{c} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -\overline{c} & 1 \end{bmatrix}$, and multiply through:

$$\begin{bmatrix} \xi^c_t \\ \widehat{b}_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\overline{c} & \beta^{-1} \end{bmatrix} \begin{bmatrix} \xi^c_{t-1} \\ \widehat{b}_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \overline{y} \end{bmatrix} \widetilde{y}_t + \begin{bmatrix} 1 \\ -\overline{c} \end{bmatrix} \eta_t.$$

Since the autoregressive coefficient matrix is triangular, the eigenvalues of the system can be read off the diagonal: $\lambda_1 = 1$, $\lambda_2 = \beta^{-1} > 1$. This matrix can be diagonalized as follows:

$$\begin{bmatrix} \frac{1-\beta}{\overline{c}\beta} & 0 \\ 1 & \beta^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \beta^{-1} \end{bmatrix} \begin{bmatrix} \frac{\overline{c}\beta}{1-\beta} & 0 \\ -\frac{\overline{c}\beta^2}{1-\beta} & \beta \end{bmatrix}.$$

Multiply the system by the matrix of right eigenvectors to get:

$$\begin{bmatrix} \frac{\bar{c}\beta}{1-\beta} & 0 \\ -\frac{\bar{c}\beta^2}{1-\beta} & \beta \end{bmatrix} \begin{bmatrix} \xi_t^c \\ \widehat{b}_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \beta^{-1} \end{bmatrix} \begin{bmatrix} \frac{\bar{c}\beta}{1-\beta} & 0 \\ -\frac{\bar{c}\beta^2}{1-\beta} & \beta \end{bmatrix} \begin{bmatrix} \xi_{t-1}^c \\ \widehat{b}_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \beta\overline{y} \end{bmatrix} \widetilde{y}_t + \begin{bmatrix} \frac{\bar{c}\beta}{1-\beta} \\ -\frac{\bar{c}\beta}{1-\beta} \end{bmatrix} \eta_t.$$

Define $w_{1t} = \frac{\bar{c}\beta}{1-\beta}\xi_t^c$ and $w_{2t} = -\frac{\bar{c}\beta^2}{1-\beta}\xi_t^c + \beta\widehat{b}_t$, then:

$$\begin{bmatrix} w_{1t} \\ w_{2t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \beta^{-1} \end{bmatrix} \begin{bmatrix} w_{1t-1} \\ w_{2t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \beta\overline{y} \end{bmatrix} \widetilde{y}_t + \begin{bmatrix} \frac{\bar{c}\beta}{1-\beta} \\ -\frac{\bar{c}\beta}{1-\beta} \end{bmatrix} \eta_t.$$

Treat $\lambda_1 = 1$ as a stable eigenvalue. Then the conditions for stability are

$$\begin{aligned} w_{2t} &= 0, \forall t, \\ \beta\overline{y}\widetilde{y}_t - \frac{\bar{c}\beta}{1-\beta}\eta_t &= 0. \end{aligned}$$

This implies a solution for the endogenous forecast error:

$$\eta_t = (1-\beta)\frac{\overline{y}}{\bar{c}}\widetilde{y}_t.$$

The decoupled system can consequently be rewritten as:

$$\begin{aligned} \begin{bmatrix} w_{1t} \\ w_{2t} \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_{1t-1} \\ w_{2t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \beta\overline{y} \end{bmatrix} \widetilde{y}_t + \begin{bmatrix} \beta\overline{y} \\ -\beta\overline{y} \end{bmatrix} \widetilde{y}_t \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_{1t-1} \\ w_{2t-1} \end{bmatrix} + \begin{bmatrix} \beta\overline{y} \\ 0 \end{bmatrix} \widetilde{y}_t. \end{aligned}$$

Now multiply by the matrix of left eigenvectors $\begin{bmatrix} \frac{1-\beta}{\bar{c}\beta} & 0 \\ 1 & \beta^{-1} \end{bmatrix}$ to return to the original set of variables:

$$\begin{bmatrix} \xi_t^c \\ \widehat{b}_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{\bar{c}\beta}{1-\beta} & 0 \end{bmatrix} \begin{bmatrix} \xi_{t-1}^c \\ \widehat{b}_{t-1} \end{bmatrix} + \begin{bmatrix} (1-\beta)\frac{\overline{y}}{\bar{c}} \\ \beta\overline{y} \end{bmatrix} \widetilde{y}_t.$$

Using the definition of ξ_t^c we find after a few steps:

$$\begin{aligned} \widetilde{c}_t &= \widetilde{c}_{t-1} + (1-\beta)\frac{\overline{y}}{\bar{c}}\widetilde{y}_t, \\ \widehat{b}_t &= \widehat{b}_{t-1} + \beta\overline{y}\widetilde{y}_t. \end{aligned}$$

The unit-root component of this model is clearly evident from the solution for consumption. Once the system is disturbed it will not return to its initial level. In fact, it will tend toward $\pm\infty$ with probability one, which raises doubts about the validity of the linearization approach in the first place. Moreover, there is no limiting distribution for the endogenous variables; the variance of consumption, for instance, is infinite. Strictly speaking, the model cannot be used for business cycle analysis.

Alternatively, one can derive the state-space representation of the solution, that is, expressed in terms of state variables and exogenous shocks. Convenient substitution thus leads to:

$$\begin{aligned}\widetilde{c}_t &= \frac{1-\beta}{\beta}\frac{\widehat{b}_{t-1}}{\overline{c}} + (1-\beta)\frac{\overline{y}}{\overline{c}}\widetilde{y}_t, \\ \frac{\widehat{b}_t}{\overline{c}} &= \frac{\widehat{b}_{t-1}}{\overline{c}} + \beta\frac{\overline{y}}{\overline{c}}\widetilde{y}_t.\end{aligned}$$

As in the intertemporal approach to the current account, income innovations only have minor affects on current consumption, but lead to substantial changes in net foreign assets. Purely temporary shocks, therefore, have permanent effects.

Bounding the Eigenvalues

The characteristic equation of a two-by-two matrix A is given by $p(\lambda) = \lambda^2 - tr\lambda + det$, where $tr = trace(A)$ and $det = \det(A)$, are the trace and determinant, respectively. According to the *Schur-Cohn Criterion* (see LaSalle 1986, 27) a necessary and sufficient condition that all roots of this polynomial be inside the unit circle is

$$|det| < 1 \quad \text{and} \quad |tr| < 1 + det.$$

I am also interested in cases in which there is one root inside the unit circle or both roots are outside the unit circle. Conditions for the former can be derived by noting that the eigenvalues of the inverse of a matrix are equal to the inverse eigenvalues of the original matrix. Define $B = A^{-1}$. Then $trace(B) = \frac{trace(A)}{\det(A)}$ and $\det(B) = \frac{1}{\det(A)}$. By Schur-Cohn, B has two eigenvalues inside the unit circle (and therefore both of A's eigenvalues are outside) if and only if $|\det(B)| < 1$ and $|trace(B)| < 1+\det(B)$. Substituting the above expressions, I find that $\left|\frac{1}{\det(A)}\right| < 1$, which implies $|\det(A)| > 1$. The second condition is $-\left(1 + \frac{1}{\det(A)}\right) < \frac{trace(A)}{\det(A)} < 1 + \frac{1}{\det(A)}$. Suppose first that $\det(A) > 0$. It follows immediately that $|trace(A)| < 1 + \det(A)$. Alternatively, if $\det(A) < 0$, I have $|trace(A)| < -(1 + \det(A))$. However, since I have restricted $|\det(A)| > 1$, the latter case collapses into the former for $\det(A) < -1$. Combining these restrictions I can then deduce that a necessary and sufficient condition for both roots lying outside the unit circle is

$$|det| > 1 \quad \text{and} \quad |tr| < 1 + det.$$

Conditions for the case of one root inside and one root outside the unit circle can be found by including all possibilities not covered by the previous ones. Consequently, I find this requires

$$\begin{aligned} \text{Either } |det| &< 1 \text{ and } |tr| > 1 + det, \\ \text{or } |det| &> 1 \text{ and } |tr| > 1 + det. \end{aligned}$$

As a side note, employing the Schur-Cohn criterion and its corollaries is preferable to using Descartes' Rule of Sign or the Fourier-Budan theorem since I may have to deal with complex eigenvalues (see Barbeau 1989, 170). Moreover, the former can give misleading bounds since it does not treat $det < -1$ as a separate restriction. This is not a problem in the canonical model where $det = \beta^{-1} > 1$, but may be relevant in the other models.

Proof of Proposition 3

With no internalization of the risk premium, the linearized equation system is given by

$$\begin{aligned} \widetilde{c}_t &= \widetilde{c}_{t-1} - \sigma\psi\widehat{b}_{t-1}, \\ \overline{c}\widetilde{c}_t + \widehat{b}_t &= R^*\left(1 - \psi\overline{b}\right)\widehat{b}_{t-1}. \end{aligned}$$

Its trace and determinant are $tr = 1 + R^*\left(1 - \psi\overline{b}\right) + \sigma\psi\bar{c}$ and $det = R^*\left(1 - \psi\overline{b}\right)$. Since I have $tr = 1 + det + \sigma\psi\bar{c} > 1 + det$, it follows immediately that the system contains one stable and one unstable root, so that the equilibrium is unique for all parameter values.

With internalization of the risk premium, the linearized equation system is given by

$$\begin{aligned} \widetilde{c}_t &= \widetilde{c}_{t-1} - \sigma\psi\left(1 + \beta R^*\right)\widehat{b}_{t-1}, \\ \overline{c}\widetilde{c}_t + \widehat{b}_t &= R^*\left(1 - \psi\overline{b}\right)\widehat{b}_{t-1}. \end{aligned}$$

Its trace and determinant are $tr = 1 + R^*\left(1 - \psi\overline{b}\right) + \sigma\psi\overline{c}\left(1 + \beta R^*\right)$ and $det = R^*\left(1 - \psi\overline{b}\right)$. Since I have $tr = 1 + det + \sigma\psi\overline{c}\left(1 + \beta R^*\right) > 1 + det$, it follows immediately that the system contains one stable and one unstable root, so that the equilibrium is unique for all parameter values. This concludes the proof of the proposition.

Proof of Proposition 4

With no internalization of the risk premium, the linearized equation system is given by

$$\begin{aligned} \widetilde{c}_t &= \widetilde{c}_{t-1} - \frac{\sigma\beta\psi}{\overline{y}}\widehat{b}_{t-1}, \\ \overline{c}\widetilde{c}_t + \widehat{b}_t &= \left(\frac{1}{\beta} - \psi\frac{\overline{b}}{\overline{y}}\right)\widehat{b}_{t-1}. \end{aligned}$$

Its trace and determinant are $tr = 1 + \sigma\beta\psi\frac{\bar{c}}{\bar{y}} + \frac{1}{\beta} - \psi\frac{\bar{b}}{\bar{y}}$ and $det = \frac{1}{\beta} - \psi\frac{\bar{b}}{\bar{y}}$. Since I have $tr = 1 + det + \sigma\beta\psi\frac{\bar{c}}{\bar{y}} > 1 + det$, it follows immediately that the system contains one stable and one unstable root, so that the equilibrium is unique for all parameter values.

With internalization of the risk premium, the linearized equation system is given by

$$\begin{aligned}\widetilde{c}_t &= \widetilde{c}_{t-1} - 2\frac{\sigma\beta\psi}{\bar{y}}\widehat{b}_{t-1},\\ \bar{c}\widetilde{c}_t + \widehat{b}_t &= \left(\frac{1}{\beta} - \psi\frac{\bar{b}}{\bar{y}}\right)\widehat{b}_{t-1}.\end{aligned}$$

Its trace and determinant are $tr = 1 + 2\sigma\beta\psi\frac{\bar{c}}{\bar{y}} + \frac{1}{\beta} - \psi\frac{\bar{b}}{\bar{y}}$ and $det = \frac{1}{\beta} - \psi\frac{\bar{b}}{\bar{y}}$. Since I have $tr = 1 + det + 2\sigma\beta\psi\frac{\bar{c}}{\bar{y}} > 1 + det$, it follows immediately that the system contains one stable and one unstable root, so that the equilibrium is unique for all parameter values. This concludes the proof of the proposition.

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